Physics Does Cartwheels: The Power of Abstract Spaces

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1 Introduction

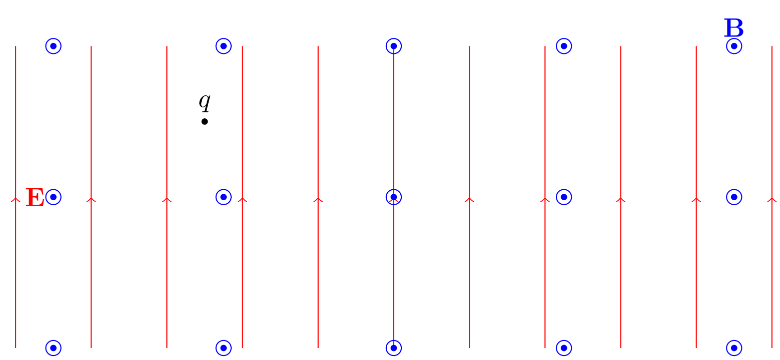
We will take a look at two solutions to a problem in electrodynamics. The first uses a more systematic method, while the second uses an elegant method — leveraging elementary geometry in an abstract velocity space. The second solution also has connections to the brachistochrone problem.
So, let's dive right in!

2 The problem

Consider a region of space with a uniform electric field **E** and a uniform magnetic field **B** of the form

$$\mathbf{E} = E\hat{\mathbf{j}} = \begin{pmatrix} 0 \\ E \\ 0 \end{pmatrix} \qquad \mathbf{B} = B\hat{\mathbf{k}} = \begin{pmatrix} 0 \\ 0 \\ B \end{pmatrix}.$$

The problem is to find the motion of a charge q in this region of space due to the Lorentz force. The diagram below illustrates the setup.



3 First Solution

For the purposes of this problem, we are only interested in 2D motion, so that the position $\mathbf{x}$ and velocity $\mathbf{v}$ are embedded in the xy plane:

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \qquad \mathbf{v} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ 0 \end{pmatrix}.$$

The Lorentz force describes the effects of $\mathbf{E}$ and $\mathbf{B}$ on the particle:

$$m\ddot{\mathbf{x}} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}$$

Using $\mathbf{E} = E\hat{\mathbf{j}}$, $\mathbf{B} = B\hat{\mathbf{k}}$, and some cross product identities, we end up with a system of differential equations for $x(t)$ and $y(t)$:

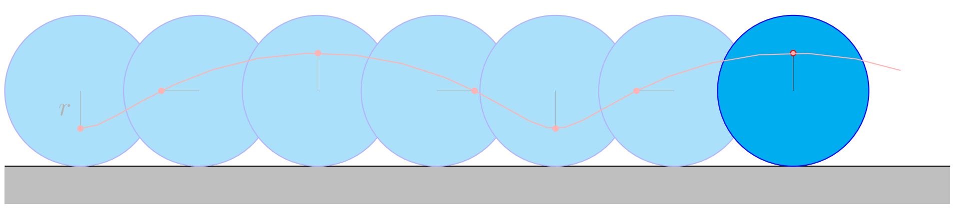
$$\ddot{x} = \frac{qB}{m}\dot{y} \tag{1}$$

$$\ddot{y} = \frac{qE}{m} - \frac{qB}{m}\dot{x}. \tag{2}$$

There are many different strategies you could use to solve this system. One thing you could do is just guess a solution for $x(t)$ and $y(t)$ - that's what I'll do here. In particular, consider the parametric equation for a trochoid:

$$\begin{cases} x &= \omega R t - r\sin\omega t \\ y &= -r\cos\omega t. \end{cases}$$

$(x(t), y(t))$ specifies the position of a point glued to a wheel rolling along the x-axis. R is the radius of the wheel, ω is the angular speed of the wheel, and r is the distance between the point and the wheel's center. The diagram below shows the rolling wheel at various times. The red trochoid curve describes the position of the red dot as a function of time.



We can identify each term in the parametric equation. The $\omega R t$ term corresponds to the x-coordinate of the center of the wheel, since ωR is the speed of the wheel due to the rolling. The trig terms are added because any point on the wheel not exactly at the center is going to be rotating around the wheel:

$$\begin{cases} x &= \underbrace{\omega R t}_{\text{translation term}} - \underbrace{r \sin \omega t}_{\text{rotation term}} \\ y &= \underbrace{-r \cos \omega t}_{\text{rotation term}} . \end{cases}$$

Anyway, the point is that the trochoid curve does actually satisfy the differential equations governing the charge's motion. We can check that this is the case by calculating the derivatives of $x(t)$ and $y(t)$,

$$\begin{cases} \dot{x} &= \omega R - \omega r \cos \omega t \\ \dot{y} &= \omega r \sin \omega t \end{cases} \qquad \begin{cases} \ddot{x} &= \omega^2 r \sin \omega t \\ \ddot{y} &= \omega^2 r \cos \omega t \end{cases};$$

and plugging them into equations (1) and (2),

$$\omega^2 r \sin \omega t = \frac{qB}{m} \omega r \sin \omega t \tag{3}$$

$$\omega^2 r \cos \omega t = \frac{qE}{m} - \frac{qB}{m} \left(\omega R - \omega r \cos \omega t \right). \tag{4}$$

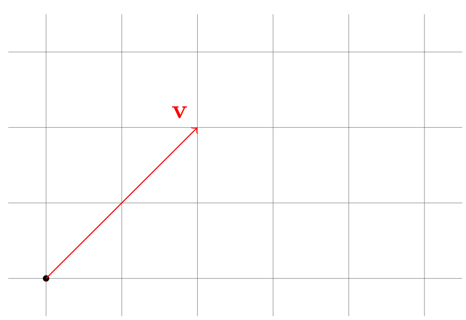
Note that the constants of motion ω and R can be read off of these expressions:

$$\omega = \frac{qB}{m} \qquad R = \frac{qE}{m\omega^2} = \frac{mE}{qB^2}.$$

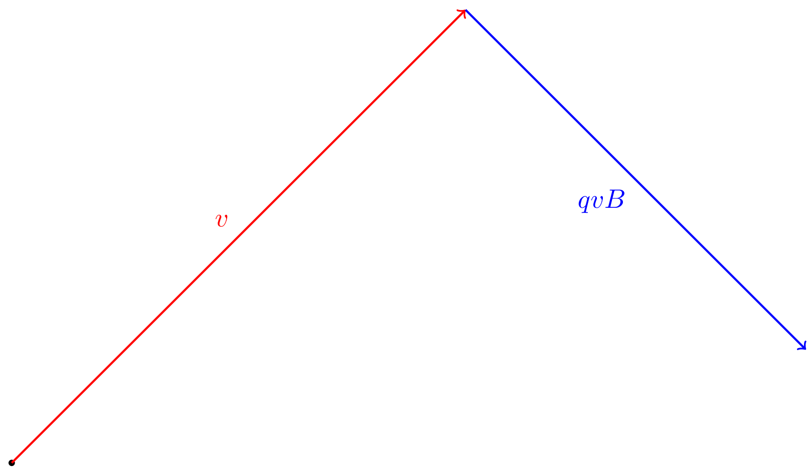
The final constant of motion r is determined by initial conditions, namely the initial velocity $\mathbf{v}(0)$.

4 Second Solution

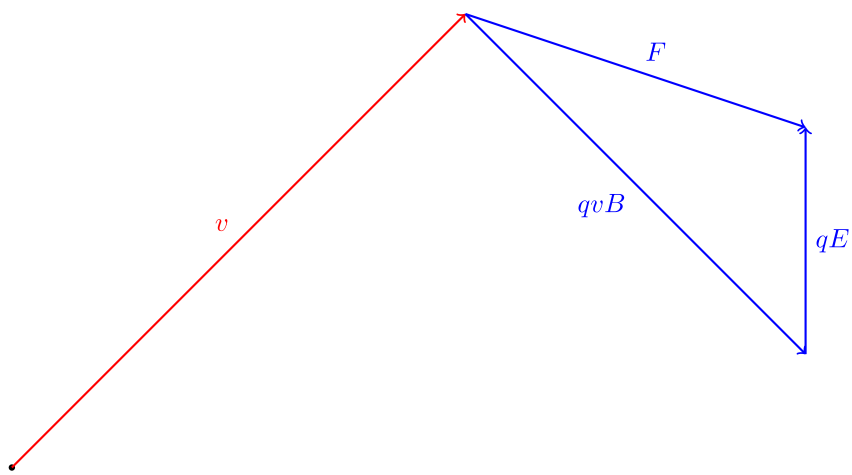
In the previous section, it was implicit that the Lorentz force is independent of position, since the fields are uniform. Indeed, the Lorentz force only depends on velocity, which we can use to our advantage by constructing a velocity space — a plane containing all possible velocity vectors $\mathbf{v}$:



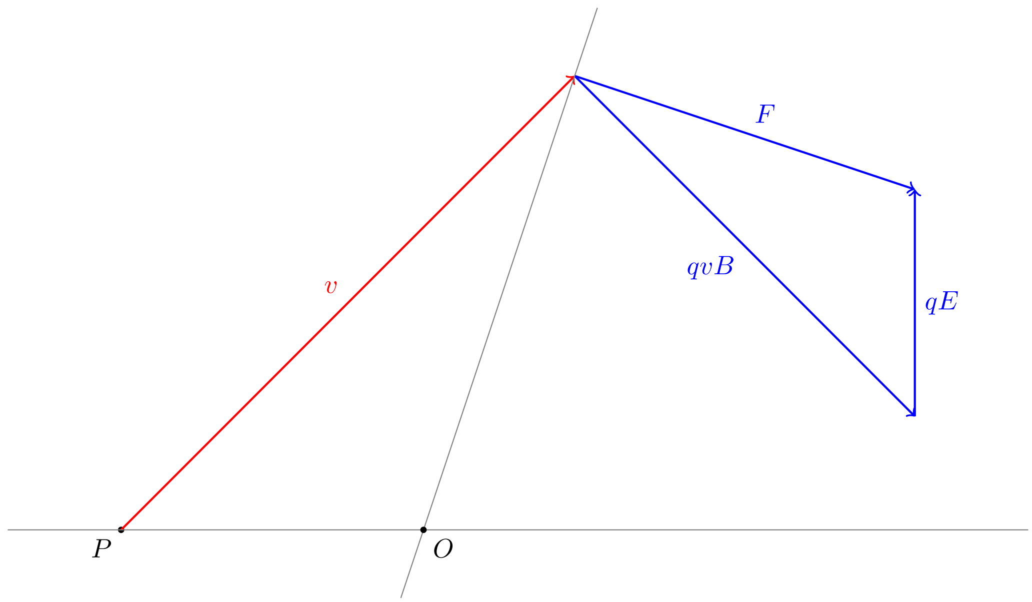
The point is that the evolution of the velocity vector only depends on where it is in the velocity space, and on no other factors. Our aim will be to predict how **v** evolves in velocity space, and to infer its trajectory in 'real' space.
So let's unpack the Lorentz force; it consists of a magnetic part perpendicular to the velocity:



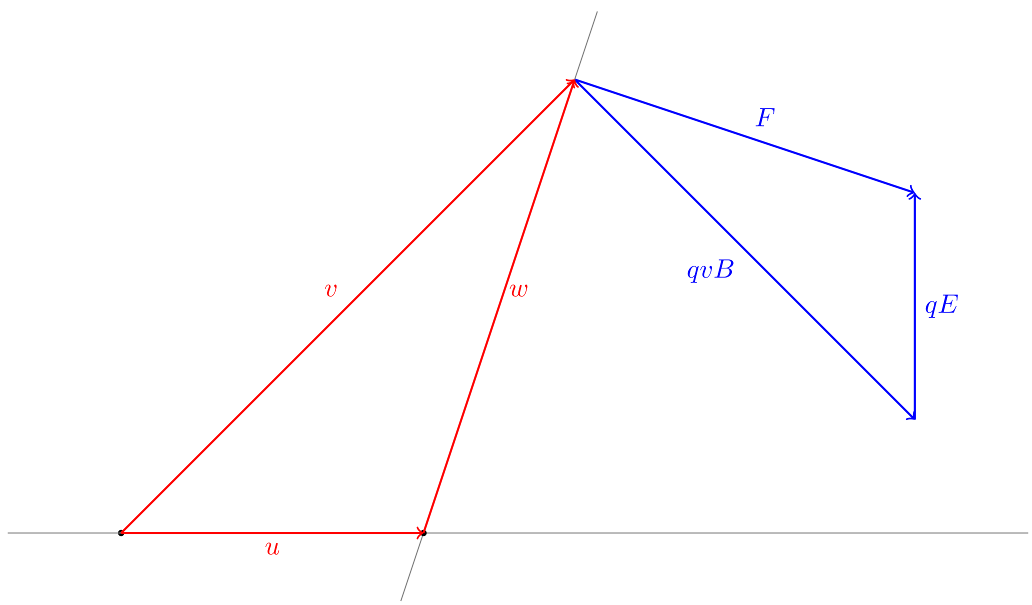
and of an electric part pointing in a constant upward direction:



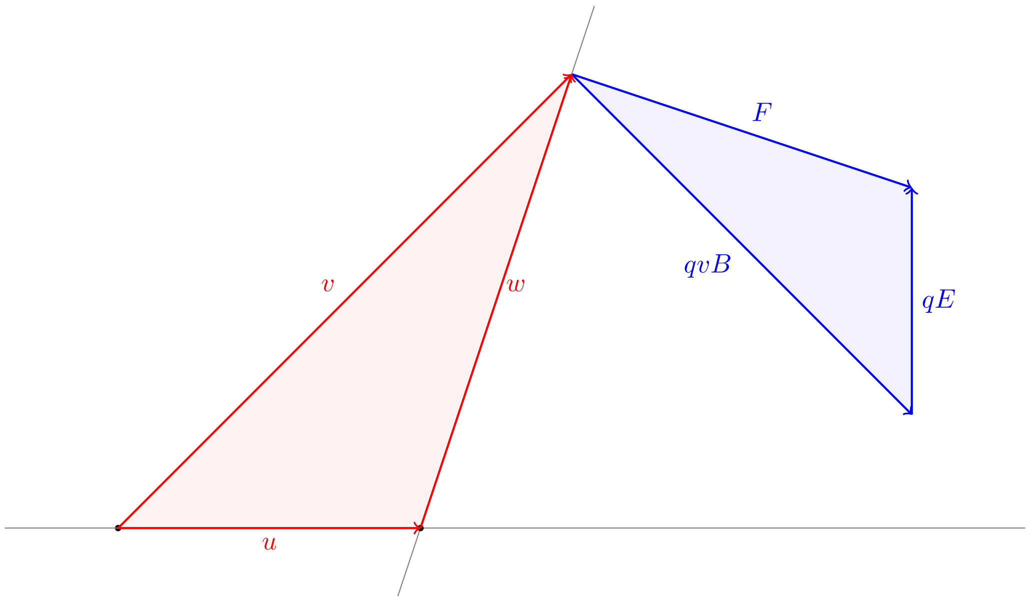
The labels next to the vectors only indicate their magnitudes. Next, introduce two new lines: one passing through the point marked P and perpendicular to the direction of the electric field, and one passing through the tip of **v** and perpendicular to **F**. Call their intersection point O:



Finally, introduce two vectors $\mathbf{u}$ and $\mathbf{w}$ aligned with the two lines such that $\mathbf{v} = \mathbf{u} + \mathbf{w}$:



It turns out that this construction leaves us with two similar triangles, offset by 90°:

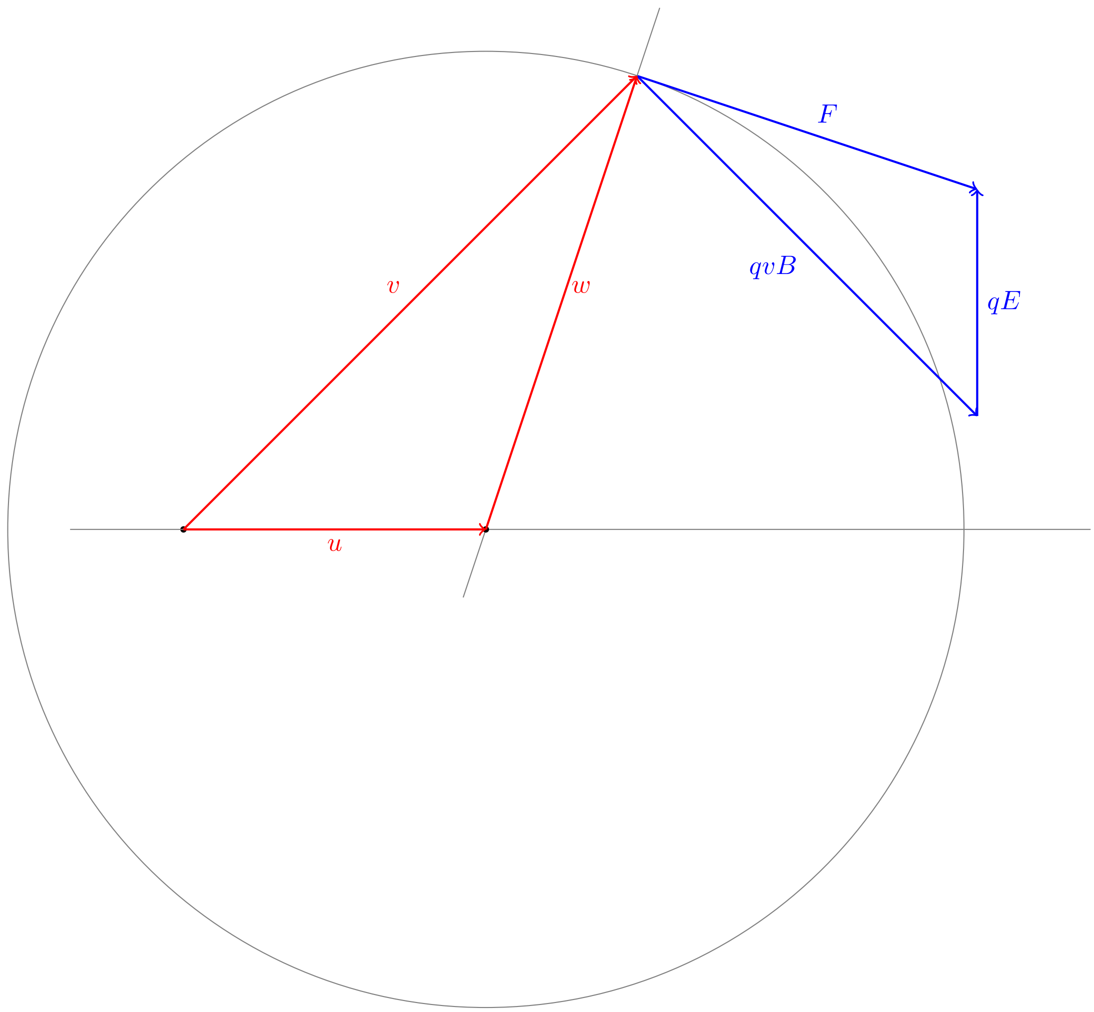


Among other things, this implies that $F \propto w$. At this point, $\mathbf{F}$ satisfies two important conditions:

1. $\mathbf{F}$ is perpendicular to $\mathbf{w}$.
2. $\mathbf{F}$ is proportional in magnitude to $\mathbf{w}$.

These two facts allow us conclude that $\mathbf{w}$ must gyrate around a circle centered at the tip of $\mathbf{u}$[1].

[1] Note that we've made the (valid) assumption that $d\mathbf{u}/dt = 0$. I've chosen not to justify it because the proof is already long enough.



Let's summarize our findings:

The dynamics of $\mathbf{v}$ are such that $\mathbf{u}$ remains fixed in place (perpendicular to $\mathbf{E}$) while $\mathbf{w}$ rotates around it anchored to its tip.

For the sake of giving names to things, let's say that ω is the angular speed of $\mathbf{w}$. This means that the Lorentz force $\mathbf{F}$ has magnitude

$$F = m\omega w.$$

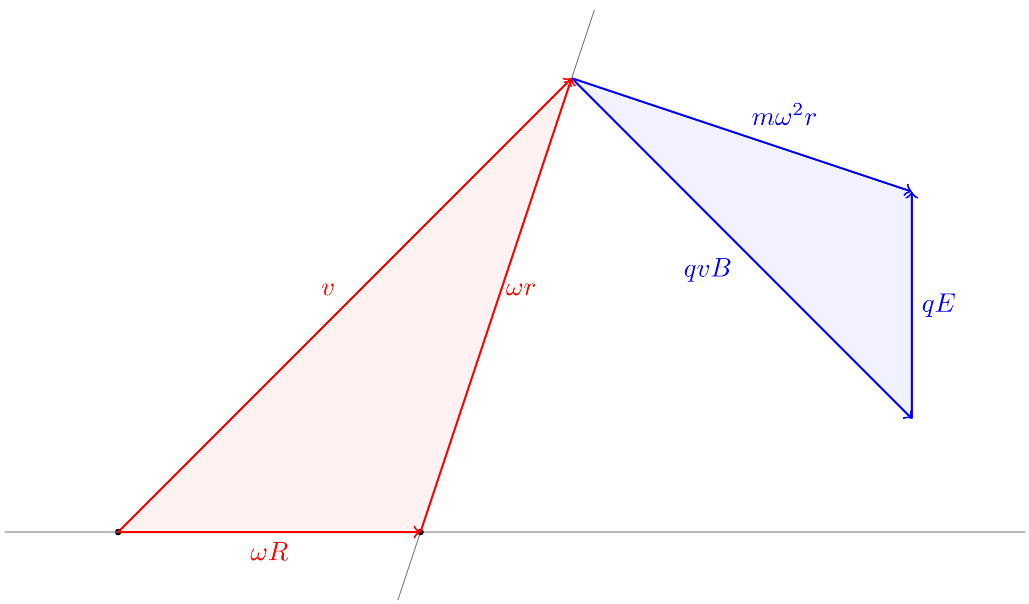
It turns out this solution of $\mathbf{v}(t)$, where $\mathbf{w}$ is spinning around a constant velocity $\mathbf{u}$, is characteristic of the velocity of a point fixed to a rolling wheel (the same wheel that we analyzed previously!). In this analogy, the wheel (of radius R rolling at angular speed ω) moves at a speed of ωR, which corresponds to u in velocity space.

$$u = \omega R.$$

A point at a distance r to the center of the wheel has an additional velocity component of size ωr, perpendicular to the wheel's radius line. This component corresponds to w in velocity space.

$$w = \omega r$$

We can also update our diagram:



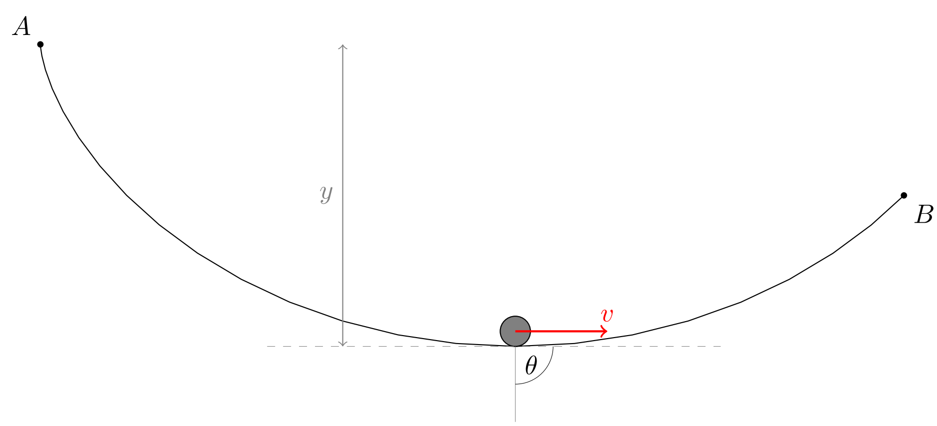
Using the fact that these are similar triangles, we can work out some useful relationships:

$$\frac{\omega r}{v} = \frac{m\omega^2 r}{qvB} \implies \omega = \frac{qB}{m}$$
$$\frac{\omega R}{\omega r} = \frac{qE}{m\omega^2 r} \implies R = \frac{qE}{m\omega^2}$$
$$\frac{\omega R}{v} = \frac{qE}{qvB} \implies \omega R = \frac{E}{B}.$$

The final of these relationships is deducible from the first two, but it's nice to see that our solution is self-consistent.

5 Relation to the Brachistochrone and 3Blue1Brown's Challenge

The brachistochrone problem is a very simple problem. How should you build a chute connecting two points A and B so that a mass released at A and sliding down travels between the two points in the shortest time possible?
It turns out that the chute should be built in the shape of a cycloid - the special case of a trochoid where $r = R$. *3Blue1Brown* presents a very neat solution to this problem in the video The Brachistochrone, with Steven Strogatz. I recommend watching it before moving on.
What I'd like to go through now is a miraculously simple solution which builds on the methods used in section 4. The solution also answers 3Blue1Brown's challenge question, an extension of the brachistochrone problem relating to the precise motion of the mass.
To start things off, it's always useful to draw a diagram containing all of the key features of the problem.

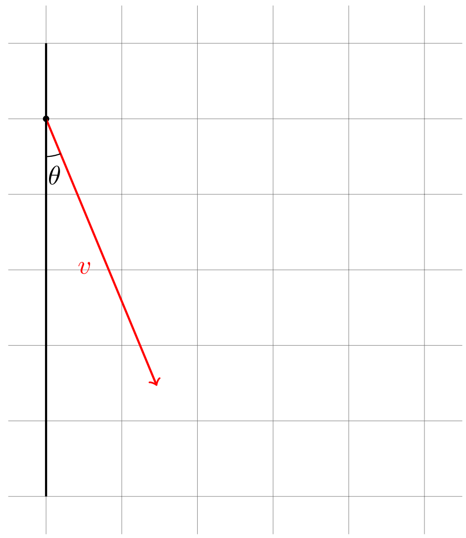


The curve joining A to B is what we'd like to find. the dashed line is tangent to the chute, and the velocity of magnitude v points in the direction of the dashed line. The velocity makes an angle θ with the vertical.
The key insight in 3Blue1Brown's presentation of the problem is that we can use Snell's law to derive an important property of the curve. Snell's law is usually applied in optics problems, but it is in fact a general result of Fermat's principle of least time. In the context of the brachistochrone problem, Snell's law tells us that

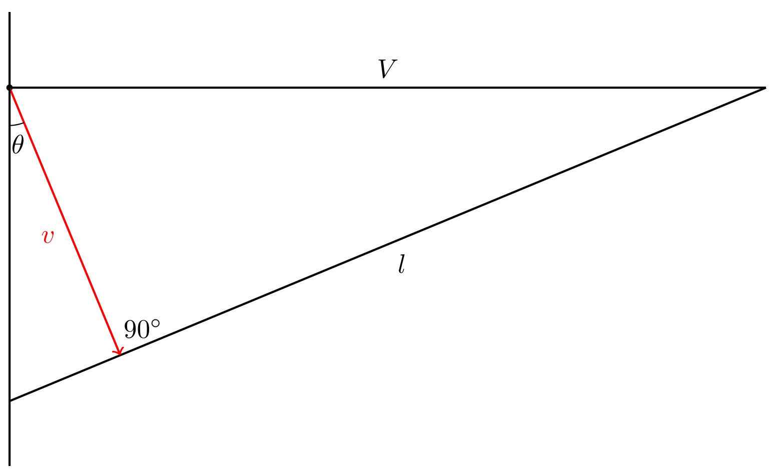
$$\boxed{\frac{\sin\theta}{v} \text{ is a constant of motion}},$$

in other words the quantity is conserved. 3Blue1Brown does a fine job of explaining this in more depth.

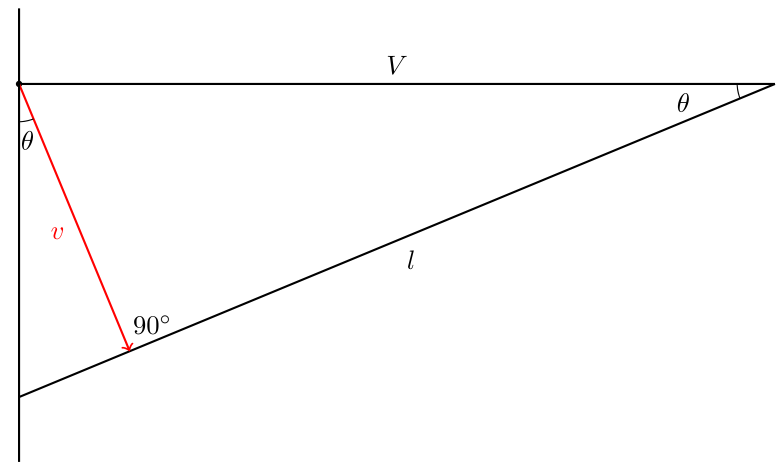
With this Snell's law property in the back of our minds, Let us think about the motion of the brachistochrone problem in velocity space. Let's say that the velocity of the mass at some point is given by the following state in velocity space:



Introduce a line l perpendicular to the velocity and passing through its tip. Then introduce a perfectly horizontal line segment V starting at the origin of velocity space and ending where it intersects l:



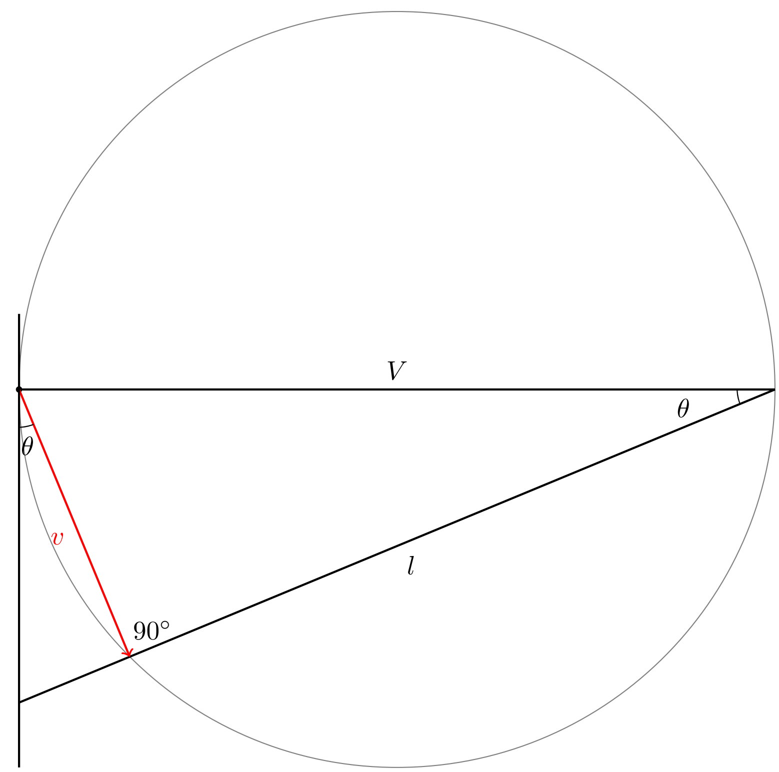
From simple angle-chasing, we find that the acute angle between l and V is θ:



Therefore,

$$v = V \sin\theta \implies V = \frac{v}{\sin\theta},$$

so V is a constant of motion! This is very significant, because V being a constant hypotenuse means that the velocity must remain on a circle of diameter V:



Finally, if the motion traces out a circle in velocity space, the trajectory and the chute must be a cycloid curve generated by a **uniformly rotating wheel**. Why must it be a uniformly rotating wheel? Well it suffices to check the parametric equations for a cycloid generated by a wheel which rolls an angle $\phi(t)$ (not necessarily equal to ωt):

$$\begin{cases} x(t) &= R\phi(t) - R\sin\phi(t) \\ y(t) &= R\cos\phi(t) \end{cases}$$

and

$$\begin{cases} \dot{x}(t) &= R\dot{\phi} - R\dot{\phi}\cos\phi \\ \dot{y}(t) &= -R\dot{\phi}\sin\phi \end{cases}$$

Unless $\dot{\phi}$ is constant, the instantaneous diameter $2\dot{\phi}R$ of the circle will change with time. We already know that the diameter of the circle is conserved, hence the generating wheel must be rolling uniformly. This completes the proof.
I think that, more than anything, these two geometric solutions show that there is often a way around the monotony of analytic methods. In closing, we leave you with the following food for thought... if a brachistochrone curve so resembles the trajectory of a charge in a uniform field, might there be equivalents of the electric and magnetic forces in the brachistochrone?